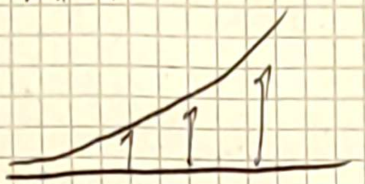
Riemannian Geometry | $\boxed{\text{Week 14}}$

Last time: We proved that if (M,g) satisfies $K \leq 0$,

then $\forall p \in M$, $\forall v, w \in T_pM$:

$$\| d\exp_p|_v (w) \| \geq \|w\|$$



Theorem (Cartan-Hadamard):

Let (M,g) be complete and connected and $K \leq 0$.

Then $\forall p \in M$, $\exp_p : T_pM \to M$ is a covering map.

Recall:

Definition: A smooth map $F: N \to M$ is a (smooth) covering map if $\forall q \in M$, $\exists$ open neighborhood U of q such that $F^{-1}(U) = \bigsqcup_i V_i$ and $\forall i$: $F: V_i \to U$ is a diffeomorphism.

Examples:
i) $\tilde{M} = M \times I \longrightarrow M$ if I is discrete

ii) $\mathbb{R}^n \longrightarrow \mathbb{T}^n$, $(x^1, \dots, x^n) \to (x^1 \bmod 1, \dots, x^n \bmod 1)$

iii) $S^n \longrightarrow P^n$,

iv) $\mathbb{C} \to \mathbb{C}^*$, $z \to e^z$

Properties of covering maps:

Let $F: \tilde{M} \to M$ be a covering map

i) If $\tilde{M}$ is connected and M simply connected

$\to F$ is a diffeo $(\tilde{M} \simeq M)$

ii) If M is connected: $\exists!$ simply connected universal cover $\tilde{M}$

And $\exists$ $G \leq \mathrm{Diff}(\tilde{M})$ acting freely and properly discontinuously on $\tilde{M}$ s.t. $\forall x, y \in \tilde{M}$:

$F(x) = F(y) \implies \exists g \in G: y = gx$

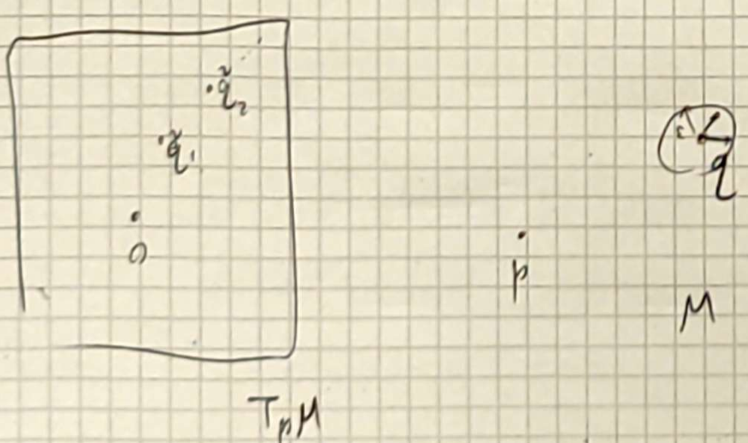
$G \simeq \pi_1(M)$

$(M = \tilde{M}/G)$

Proof of Cartan-Hadamard:

Since $d\exp_p|_v$ doesn't have a kernel $\forall v \in T_pM$: $\exp_p$ is a local diffeo. $\exp_p$ is also onto (by Hopf-Rinow)

Let $q \in M$ and let $\exp_p^{-1}(q) = \{\tilde{q}_i\}_{i \in I}$.



~~Note: Since $\exp_p$ is a local diffeo. Every curve $\gamma: [0,1] \to M$ has a~~
~~unique lift $\tilde{\gamma}: [0,1] \to T_pM$ s.t. $\exp_p(\tilde{\gamma}(t)) = \gamma(t)$ (because locally fixed $\tilde{\gamma}(0)$)~~

Denote $F = \exp_p$ (for simplicity)

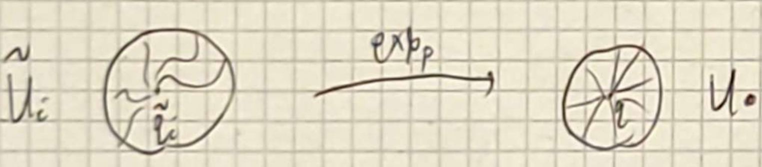
If $\gamma : [0,1] \to M$ is a smooth curve, $q = \gamma(0)$ and $\tilde{q} \in T_pM$ is such that $F(\tilde{q}) = q$, then $\exists !$ smooth curve $\tilde{\gamma} : [0,1] \to T_pM$ such that $\gamma = F(\tilde{\gamma})$.

<u>Proof:</u> ~~...~~ If $\tilde{\gamma} : [0, t_0] \to T_pM$ is ~~...~~ such a curve with $t_0 < 1$: I can extend to $[0, t_0 + \delta)$ for some $\delta > 0$ using the fact that F is a local diffeo on a neighborhood of $\tilde{\gamma}(t_0)$ (openness)

~~...~~ If $\tilde{\gamma}$ is defined on $[0, t_0)$, then it can be extended to $[0, t_0]$ (closeness) using the fact that $\ell(\tilde{\gamma}|_{[0,t_0)}) \leq \ell(\gamma|_{[0,t_0)}) \leq \ell(\gamma)$ (following from the bound $\| d\exp_p|_v(w) \| \geq \|w\|$). ☐

Let $\varepsilon > 0$ be small enough so that $U = B_\varepsilon(q)$ lies inside the domain of a normal coordinate chart around q. U is a union of normal geodesic segments emanating from q

$\Rightarrow$ By lifting those segments to curves in T_pM starting from $\tilde{q}_i$ I get open neighborhoods $\tilde{U}_i$ of $\tilde{q}_i$

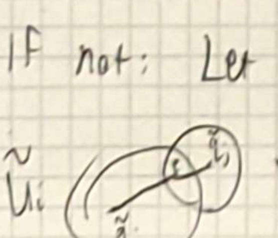


Note: $\exp_p : \tilde{U}_i \to U$ is a diffeomorphism

(it is a local diffeo, it is onto (by definition) and it is 1-1, since otherwise if $\exp_p(x) = \exp_p(y)$ for $x \neq y$, I would get that the corresponding radial geodesics reintersect in U, which cannot happen)

So: It suffices to prove that $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ if $i \neq j$.

If not: Let $z \in \tilde{U}_i \cap \tilde{U}_j$



Then $\exists$ curve $\tilde{\sigma} : \tilde{q}_i \to z \to \tilde{q}_j$ such that $\tilde{\sigma} \subset \tilde{U}_i \cup \tilde{U}_j$

$\Rightarrow \sigma = \exp_p(\tilde{\sigma})$ is a <u>closed loop</u> inside U

But U is simply connected, so σ can be continuously deformed to q

$\Longrightarrow$ The deformation can be lifted to T_pM

This is a contradiction if $q_i \neq q_j$, since a small enough loop $q \to q$ will have to lift to a closed loop in T_pM (since $\exp_p$ is a local diffeo around q_i) $\boxed{\times}$

Corollary:

If (M,g) is connected, complete and $K \leq 0$, then:

i) If M is simply connected, then $M \simeq \mathbb{R}^n$.

ii) In general, the universal cover of M is $\simeq \mathbb{R}^n$

iii) If M is compact, then $|\pi_1(M)| = \infty$.

So: Does S^n admit a metric with $K \leq 0$? NO

(But for $n \geq 3$, we can always find a metric with $Ric \leq 0$)

Corollary:

With (M,g) as above:

i) If $K = 0$, then $\exists\, G \subseteq Isom(\mathbb{R}^n)$ s.t $M \simeq \mathbb{R}^n/G$

ii) If $K = -1$, $\exists\, G \subseteq Isom(\mathbb{H}^n)$ s.t $M \simeq \mathbb{H}^n/G$.

(In both cases: G acts freely and properly discontinuously).

Exam: I will send an email to arrange a meeting a few days before for Q & As.

Material: do Carmo: Chapters 1-7, 9.3
(plus vector field flows).

- Smooth manifolds, tangent - cotangent space, vector fields, tensor fields
- Riem. metrics, lengths of curves, distance.
  Proof: Any smooth manifold admits a Riem. metric
- Christoffel symbols, geodesics, connections
  Proof: Existence - uniqueness of Levi-Civita connection
- Parallel transport
- Normal coordinates, Gauss-lemma, polar coordinates in 2d
- Hopf - Rinow (outline of the proof)
- Riemann curvature, Ricci tensor, scalar curvature
  (symmetries..)
- Submanifolds
- 1^{st} and 2^{nd} variation formulas for length (and proof)
- Bonnet - Myers, Jacobi vector fields
- Cartan - Hadamard (sketch of proof)

Also: Some theory in the exercises: Killing vector fields, naturality of the exponential map, preservation via isometries of ∇, R, etc

Calculation of pullback metrics